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Six-Dimensional Formulation of Meson Equations

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Abstract

The structure of the conformal group is studied by a generalisation of quaternion methods to six dimensions. Some simple $SO(4,2)$ covariant equations are shown to correspond to the Kemmer formulation for pseudoscalar and vector mesons, and the matrix elements of the irreducible representations of the Kemmer algebra are expressed as traces of products of Dirac matrices.

1. *Introduction*

There has been a recent renewed interest in the conformal group (Mack & Salam, 1969; Barut, 1968) and the de-Sitter group (Bakri, 1969; Bakri *et al.*, 1970) for the formulation of physical theories. We shall study the structure of the spinor representations of the group $SO(4,2)$ by means of the algebraic methods introduced in a previous paper (Lord, 1971), and construct simple $SO(4,2)$ covariant equations which correspond, in a Minkowzki subspace, to the Dirac equation for the electron and the Kemmer equations for scalar and vector mesons (Kemmer, 1939, 1943). This leads to an expression for the matrix elements of the irreducible representations of the four- and five-dimensional Kemmer algebras as traces of products of Dirac matrices.

Let α^μ $(\mu = 1,\dots 4)$ be a four-dimensional irreducible representation of the generators of the Dirac algebra

$$\alpha^{(\mu}\alpha^{\nu)} = -g^{\mu\nu} \tag{1.1}$$

where $g^{\mu\nu}$ is the diagonal flat-space metric $(+++-)$ used to raise and lower four-fold vector indices. Here and in the following work a pair of round brackets enclosing an index set will denote symmetrisation, and square brackets will denote skew-symmetrisation.

$$\alpha^5 = i\alpha^1\alpha^2\alpha^3\alpha^4 = (i/24)\,\epsilon_{\mu\nu\rho\sigma}\,\alpha^\mu\alpha^\nu\alpha^\rho\alpha^\sigma$$

has unit square and anti-commutes with the α^μ. The 15 traceless base elements of Dirac's algebra are infinitesimal generators of $SO(4,2)$ (ten

Kate, 1968). The main purpose of the present work is to display the properties of the algebra in a notation which will enable full use to be made of this fact, and to construct manifestly covariant $SO(4,2)$ tensor and spinor equations.

We define the six quantities

$$\sigma^A = (i\alpha^5\alpha^\mu, \alpha^5, 1) \qquad (A = 1, \ldots 6) \tag{1.2}$$

and the 'conjugated' quantities

$$\bar{\sigma}^A = (-i\alpha^5\alpha^\mu, -\alpha^5, 1) \tag{1.3}$$

We then find

$$\left.\begin{aligned} \sigma^{(A}\,\bar{\sigma}^{B)} &= g^{AB} \\ \bar{\sigma}^{(A}\,\sigma^{B)} &= g^{AB} \end{aligned}\right\} \tag{1.4}$$

where g^{AB} is the diagonal matrix $(++++--+)$, the metric of $SO(4,2)$ which will be used as a raising and lowering operator for the six-fold indices. Defining

$$\sigma^{AB} = \sigma^{[A}\,\bar{\sigma}^{B]}, \qquad \bar{\sigma}^{AB} = \bar{\sigma}^{[A}\,\sigma^{B]} \tag{1.5}$$

these quantities satisfy

$$\left.\begin{aligned} [\sigma^{AB}, \sigma^{CD}] &= 2(g^{BC}\,\sigma^{AD} - g^{AC}\,\sigma^{BD} + g^{AD}\,\sigma^{BC} - g^{AD}\,\sigma^{BC}) \\ [\bar{\sigma}^{AB}, \bar{\sigma}^{CD}] &= 2(g^{BC}\,\bar{\sigma}^{AD} - g^{AC}\,\bar{\sigma}^{BD} + g^{AD}\,\bar{\sigma}^{BC} - g^{AD}\,\bar{\sigma}^{BC}) \end{aligned}\right\} \tag{1.6}$$

so that $\sigma^{AB}/2$ and $\bar{\sigma}^{AB}/2$ are generators of representations S and $\bar{S}$ of $SO(4,2)$. We also have

$$\left.\begin{aligned} \tfrac{1}{2}(\sigma^{BC}\,\sigma^A - \sigma^A\,\bar{\sigma}^{BC}) &= g^{AC}\,\sigma^B - g^{BC}\,\sigma^A \\ \tfrac{1}{2}(\bar{\sigma}^{BC}\,\bar{\sigma}^A - \bar{\sigma}^A\,\sigma^{BC}) &= g^{AC}\,\bar{\sigma}^B - g^{BC}\,\bar{\sigma}^A \end{aligned}\right\} \tag{1.7}$$

which are the infinitesimal forms of the statement that, if O_A^B is a matrix of $SO(4,2)$ and S and $\bar{S}$ its 4×4 matrix representations described above (which we shall call the 'basic' spinor representations), then

$$O_B^A\, S\sigma^B\,\bar{S}^{-1} = \sigma^A, \qquad O_B^A\,\bar{S}\bar{\sigma}^B\,S^{-1} = \bar{\sigma}^A \tag{1.8}$$

We also define

$$\sigma^{ABC} = \sigma^{[A}\,\bar{\sigma}^B\,\sigma^{C]}, \qquad \bar{\sigma}^{ABC} = \bar{\sigma}^{[A}\,\sigma^B\,\bar{\sigma}^{C]} \tag{1.9}$$

which satisfy the self-dual and anti-self-dual properties

$$\left.\begin{aligned} \sigma^{ABC} &= (i/6)\,\epsilon^{ABCDEF}\,\sigma_{DEF} \\ \bar{\sigma}^{ABC} &= -(i/6)\,\epsilon^{ABCDEF}\,\bar{\sigma}_{DEF} \end{aligned}\right\} \tag{1.10}$$

where ϵ_{ABCDEF} is the completely skew-symmetric $SO(4,2)$ tensor with $\epsilon_{123456} = 1$. The general theory for N dimensions, in which the above relations relate to the special case $N = 6$, are given in previous work (Lord, 1971). The following relations will also be found useful. To save space they are stated without proof, but are easily verified. By expressing them in terms of Dirac matrices by (1.2) and (1.3) we obtain well-known relations for the Dirac algebra.

$$\left.\begin{aligned}\{\sigma^{AB},\sigma^{CD}\} &= \sigma^{AB}\sigma^{CD}+\sigma^{CD}\sigma^{AB} = i\epsilon^{ABCDEF}\sigma_{EF}+2(g^{BC}g^{AD}-g^{AC}g^{BD})\\ \{\bar\sigma^{AB},\bar\sigma^{CD}\} &= -i\epsilon^{ABCDEF}\bar\sigma_{EF}+2(g^{BC}g^{AD}-g^{AC}g^{BD})\end{aligned}\right\}\quad(1.11)$$

$$\left.\begin{aligned}\sigma^{A}\bar\sigma^{BCD} &= (i/2)\,\epsilon^{ABCDEF}\sigma_{EF}+g^{AB}\sigma^{CD}+g^{AC}\sigma^{DB}+g^{AD}\sigma^{BC}\\ \bar\sigma^{A}\sigma^{BCD} &= -(i/2)\,\epsilon^{ABCDEF}\bar\sigma_{EF}+3g^{A[B}\bar\sigma^{CD]}\end{aligned}\right\}\quad(1.12)$$

$$\left.\begin{aligned}&\tfrac{1}{4}\,\text{trace}\,\sigma^{AB}\sigma_{CD}=-2\delta^{AB}_{[CD]},\qquad \tfrac{1}{4}\,\text{trace}\,\sigma^{A}\bar\sigma_{B}=\delta^{A}_{B}\\ &\tfrac{1}{4}\,\text{trace}\,\sigma^{A}\bar\sigma_{BCD}=0,\qquad \tfrac{1}{4}\,\text{trace}\,\sigma^{ABC}\bar\sigma_{DEF}=i\epsilon^{ABC}_{DEF}-6\delta^{ABC}_{[DEF]}\\ &\tfrac{1}{4}\,\text{trace}\,\sigma^{AB}=\tfrac{1}{4}\,\text{trace}\,\bar\sigma^{AB}=0\end{aligned}\right\}\quad(1.13)$$

$$\left.\begin{aligned}\bar\sigma^{A}\sigma^{BC} &= \bar\sigma^{ABC}+g^{AB}\bar\sigma^{C}-g^{AC}\bar\sigma^{B}\\ \sigma^{BC}\sigma^{A} &= \sigma^{BCA}+g^{AC}\sigma^{B}-g^{AB}\sigma^{C}\end{aligned}\right\}\quad(1.14)$$

2. *Abstract Formulation*

It should be noted that the relations we have obtained, could be regarded as the basis for a theory of generalised quaternions without reference to the particular four-dimensional representations (1.2). Hamilton's quaternions in fact were formulated as an abstract algebra before the two-dimensional irreducible representations (Pauli matrices) were known. In this context

$$\left.\begin{aligned}\sigma_A &= (\sigma_a, 1)\qquad (a=1,\ldots 5)\\ \bar\sigma_A &= (-\sigma_a, 1)\end{aligned}\right\}\quad(2.1)$$

together with (1.4) are defining relations for the algebra. We can distinguish four distinct types of general element of the algebra (this is so also in Hamilton's quaternions, but this has been obscured by the fact that the algebra was evolved to deal with three dimensions, in which the distinction between a vector and a rank two skew-symmetric tensor is obscured). This has led to a certain amount of confusion in the literature in the applications of quaternions to four-dimensional problems (special relativity). For a discussion of the nature of the difficulties involved, the reader is referred to the work of Ellis (1966).

We can expand a general element of the algebra in any of four ways:

$$\left.\begin{aligned}&(a)\ \Xi=\xi_A\sigma^A+(1/12)\,\xi_{ABC}\sigma^{ABC}\\ &(b)\ \Xi=\bar\xi_A\bar\sigma^A+(1/12)\,\bar\xi_{ABC}\bar\sigma^{ABC}\\ &(c)\ \Xi=\xi+(1/2)\,\xi_{AB}\sigma^{AB}\\ &(d)\ \Xi=\bar\xi+(1/2)\,\bar\xi_{AB}\bar\sigma^{AB}\end{aligned}\right\}\quad(2.2)$$

The factor (1/12) in (a) and (b) is inserted because in the summation each term occurs 12 times.

$\bar\xi_{ABC}$ is a self-dual tensor and ξ_{ABC} is anti-self-dual. Under an inversion in the five space perpendicular to σ^6 we have $\xi_A \leftrightarrow \bar\xi_A$, $\xi_{ABC} \leftrightarrow \bar\xi_{ABC}$, $\xi_{AB} \leftrightarrow \bar\xi_{AB}$.

The tensors of various ranks are the generalisations of Hamilton's 'scalar' and 'vector' part of a quaternion. Using Hamilton's notation we may write $\xi = S\Xi$ for the scalar part. In the abstract approach to generalised quaternions the operator S takes the place of $\frac{1}{4}$ trace in (1.13). It is our aim to use the algebra to deal with rotations in six-dimensions, as an extension of the quaternion methods of special relativity (Rastall, 1964). When this is done we are no longer free to decompose Ξ in any one of the four ways (2.2), since each particular way will correspond to a different transformation law for Ξ (equation (1.8) should clarify this statement).

3. *The Adjoint Spinor and the Charge-Conjugation Matrix*

It easily follows, from the analogous well-known results for the matrices α^μ that there exists a matrix β with the following properties:

$$\beta = \beta^\dagger = \beta^{-1} \tag{3.1}$$

$$\left.\begin{aligned} \beta\sigma^A\beta &= (\bar{\sigma}^A)^\dagger \\ \beta\bar{\sigma}^A\beta &= (\sigma^A)^\dagger \end{aligned}\right\} \tag{3.2}$$

so that

$$\beta\sigma^{AB}\beta = -(\sigma^{AB})^\dagger, \qquad \beta\bar{\sigma}^{AB}\beta = -(\bar{\sigma}^{AB})^\dagger \tag{3.3}$$

which is the infinitesimal form of

$$\beta S\beta = (S^\dagger)^{-1}, \qquad \text{or} \qquad S^\dagger\beta = \beta S^{-1} \tag{3.4}$$

This shows that if ψ is a 'basic spinor' of $SO(4,2)$ (transforming to $S\psi$ under the $SO(4,2)$ rotation O_A^B), then $\psi^\dagger\beta = \bar{\psi}$ transforms to $\bar{\psi}S^{-1}$ so that $\bar{\psi}\psi$ is invariant. Since β has two eigenvalues +1 and two eigenvalues −1, this just corresponds to the well-known fact that the universal covering group of $SO(4,2)$ is $SU(2,2)$.

The equations

$$\left.\begin{aligned} S\beta S^\dagger &= \beta, & S^{-1}\beta(S^\dagger)^{-1} &= \beta \\ S^\dagger\beta S &= \beta, & (S^\dagger)^{-1}\beta S^{-1} &= \beta \\ \bar{S}\beta\bar{S}^\dagger &= \beta, & \bar{S}^{-1}\beta(\bar{S}^\dagger)^{-1} &= \beta \\ \bar{S}^\dagger\beta\bar{S} &= \beta, & (\bar{S}^\dagger)^{-1}\beta\bar{S}^{-1} &= \beta \end{aligned}\right\} \tag{3.5}$$

which are obtained from (3.3) and (3.4) can be regarded as alternative $SO(4,2)$ transformation laws of β under which it remains invariant.

A particular consequence of (3.4) is that the matrices $\sigma^{AB}\beta$ are all Hermitian. Any (4 × 4) Hermitian matrix can therefore be written

$$\left.\begin{aligned} &\Xi = (\xi + \tfrac{1}{2}\xi_{AB}\sigma^{AB})\beta \\ &\xi = \tfrac{1}{4}\,\text{trace}\,\Xi\beta, \qquad \xi_{AB} = -\tfrac{1}{4}\,\text{trace}\,\Xi\beta\sigma_{AB} \end{aligned}\right\} \tag{3.6}$$

where the coefficients ξ and ξ_{AB} are real. Moreover, if Ξ is assigned the transformation law $\Xi \to S\Xi S^\dagger$ it is easily proved, from (1.8) and the first equation (3.5), that ξ is an invariant and ξ_{AB} transforms as a tensor.

Another matrix of importance is the charge-conjugation matrix C (Corson, 1954) for the α^μ matrices. It satisfies

$$C = -C^T = -C^{-1} \tag{3.7}$$

$$C\sigma^A C = -(\sigma^A)^T, \qquad C\bar{\sigma}^A C = -(\bar{\sigma}^A)^T \tag{3.8}$$

This gives

$$C\sigma^{AB} C = (\bar{\sigma}^{AB})^T \tag{3.9}$$

which is the infinitesimal form of

$$CS^{-1} = \bar{S}^T C \tag{3.10}$$

Thus if ψ and χ are two basic spinors, transforming to $S\psi$ and $\bar{S}\chi$ respectively, then

$$\chi^T C\psi \tag{3.11}$$

is an $SO(4,2)$ invariant. It is a simplectic form in the components of ψ and χ. We have in (3.11) a six-dimensional analogue of the property of two-component spinor representations ϕ and ζ of the Lorentz group, that $\phi_1 \zeta_2 - \phi_2 \zeta_1$ is invariant.

A consequence of (3.8) is that the matrices

$$\left.\begin{array}{ll} \sigma^A C & \text{are all skew-symmetric} \\ \sigma^{ABC} C & \text{are all symmetric} \end{array}\right\} \tag{3.12}$$

So that a skew-symmetric 4×4 matrix can be written

$$\Xi = \tfrac{1}{2}\xi_A \sigma^A C; \qquad \xi_A = -\tfrac{1}{4} \operatorname{trace} \Xi C\bar{\sigma}_A \tag{3.13}$$

and a symmetric 4×4 matrix can be written

$$\Xi = \tfrac{1}{12}\xi_{ABC} \sigma^{ABC} C; \qquad \xi_{ABC} = \tfrac{1}{4} \operatorname{trace} \Xi C\bar{\sigma}_{ABC} \tag{3.14}$$

Moreover, if these Ξ are assigned the transformation law $\Xi \to S\Xi S^T$ then the components (3.13) transform as a six-vector and those of (3.14) as a (self-dual) skew-symmetric tensor of rank 3. For completeness, we list for as we did for B, the different $SO(4,2)$ transformation laws for C that will leave its components invariant.

$$\left.\begin{array}{ll} SC\bar{S}^T = C, & S^{-1} C(\bar{S}^T)^{-1} = C \\ \bar{S}^T CS = C, & (\bar{S}^T)^{-1} CS^{-1} = C \\ \bar{S}CS^T = C, & (\bar{S})^{-1} C(S^T)^{-1} = C \\ S^T C\bar{S} = C, & (S^T)^{-1} C\bar{S}^{-1} = C \end{array}\right\} \tag{3.15}$$

4. *Spinor Indices*

So far we have got by without writing spinor indices explicitly, by using a matrix notation. We now introduce different kinds of indices for different transformation laws, as in the two-component spinor algebra of the Lorentz group.

$$\left.\begin{array}{ll}\text{(a)} & \psi_\alpha \text{ indicates the transformation law } \psi \to S\psi \\ \text{(b)} & \psi^\alpha \text{ indicates the transformation law } \psi \to \psi S^{-1} = (S^{-1})^T \psi \\ \text{(c)} & \chi_{\bar{\alpha}} \text{ indicates the transformation law } \chi \to \chi \bar{S}^{-1} = (\bar{S}^{-1})^T \chi \\ \text{(d)} & \chi^{\bar{\alpha}} \text{ indicates the transformation law } \chi \to \bar{S}\chi = \chi \bar{S}^T\end{array}\right\} \tag{4.1}$$

In addition we will use dotted indices when S is replaced by its complex conjugate in (4.1). A barred and dotted index, for notational simplicity, can be written as a primed index. Thus, for example

$$\chi^{\alpha'} \to \chi^{\beta'}(\bar{S}^\dagger)_{\beta'}^{\alpha'} \tag{4.2}$$

The assignments of indices to the various matrices we have defined follows immediately from our previous discussion. Thus we have

$$\sigma^A_{\alpha\bar{\beta}}, \qquad \bar{\sigma}^{\bar{\alpha}\beta}_A \tag{4.3}$$

to comply with (1.8), and from (3.5) the matrix β can be assigned indices in any one of the following ways

$$\beta_{\alpha\dot{\beta}},\ \beta_{\dot{\alpha}\beta},\ \beta^{\dot{\alpha}\beta},\ \beta^{\alpha\dot{\beta}},\ \beta_{\bar{\alpha}\beta'} \text{ etc.} \tag{4.4}$$

and from (3.5) the possible assignments for C are

$$C_\alpha{}^{\bar{\beta}},\ C^{\bar{\beta}}{}_\alpha,\ C^\alpha{}_{\bar{\beta}},\ C_{\bar{\alpha}}{}^\beta,\ C_{\alpha'}{}^{\dot{\beta}} \text{ etc.} \tag{4.5}$$

All the formalism we have now evolved has achieved the following simple prescription for constructing manifestly-covariant $SO(4,2)$ spinor equations. We simply adhere to the rule that summation over a pair of indices can be carried out only if the two indices are of the *same type*, one as a subscript and one as a superscript.

Note that, although there is no raising and lowering operator for spinor indices of $SO(4,2)$, we can use β and C to convert an index from one kind to another, for instance, given ψ_α we can define

$$C_\alpha{}^{\bar{\beta}}\,\psi_{\bar{\beta}} = \psi_\alpha, \qquad \psi_\beta\, C^\beta{}_{\bar{\alpha}} = \psi_{\bar{\alpha}} \tag{4.6}$$

Note that care must be taken to achieve consistency when C is used in this way, on account of its skew-symmetry.

A raising and lowering operator does exist for *skew-symmetric pairs* of spinor indices. This is the completely skew spinor $\epsilon_{\alpha\beta\gamma\delta}$ ($\epsilon_{1234} = 1$). Its invariance is a consequence of the unimodularity of S. We use C to define the following sets of skew-symmetric matrices:

$$\left.\begin{array}{l}\sigma^A_{\alpha\beta} = \sigma^A_{\alpha\bar{\beta}}\, C_\beta{}^{\bar{\beta}} \\ \sigma^{\alpha\beta}_A = C^\alpha{}_{\bar{\alpha}}\, \bar{\sigma}^{\bar{\alpha}\beta}_A\end{array}\right\} \tag{4.7}$$

(Note that if we are explicitly writing in the spinor indices, the bar on $\bar{\sigma}$ can be omitted without ambiguity). We then have the curious $SO(4,2)$ identity

$$\sigma^A_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\,\sigma^{A\gamma\delta} \tag{4.8}$$

Further relations, which follow from the symmetry of $\sigma_{\alpha\beta}^{ABC}$, the skew-symmetry of $\sigma_{\alpha\beta}^{A}$ and the traceless property of $\sigma^{AB}{}_{\alpha}^{\beta}$ (in their spinor indices), the fact that either σ^A, σ^{ABC} or $1, \sigma^{AB}$ give complete linearly independent bases for the expansion of any 4×4 matrix, are the following:

$$\tfrac{1}{2}\sigma_{AB}{}_{\beta}^{\alpha}\,\sigma^{AB}{}_{\delta}^{\gamma} = (\delta_{\beta}^{\alpha}\,\delta_{\delta}^{\gamma} - 4\delta_{\beta}^{\gamma}\,\delta_{\delta}^{\alpha}) \tag{4.9}$$

$$\tfrac{1}{12}\sigma_{ABC}^{\alpha\beta}\,\sigma_{\gamma\delta}^{ABC} = -2(\delta_{\gamma}^{\alpha}\,\delta_{\delta}^{\beta} + \delta_{\delta}^{\alpha}\,\delta_{\gamma}^{\beta}) \tag{4.10}$$

$$\sigma_{A}^{\alpha\beta}\,\sigma_{\gamma\delta}^{A} = 2(\delta_{\delta}^{\alpha}\,\delta_{\gamma}^{\beta} - \delta_{\gamma}^{\alpha}\,\delta_{\delta}^{\beta}) \tag{4.11}$$

5. *Covariant Equations*

Given a six-vector p_A, we can define from it a rank 2 spinor

$$\bar{P} = p_A\,\bar{\sigma}^A \tag{5.1}$$

The simple $SO(4,2)$ invariant equation

$$\bar{P}\psi = 0 \tag{5.2}$$

can be constructed from the vector and a basic spinor ψ. Multiplying by $P = p_A\,\sigma^A$ and using (1.4) we have

$$(p_A p^A)\,\psi = 0 \tag{5.3}$$

Equation (5.2) can be regarded as an analogue in six-dimensions of the Weyl equation for a massless spin-half particle. In the particular frame in which $p_6 = 0$, the equation (5.2) takes on the form of a Dirac equation if p_μ ($\mu = 1, \ldots 4$) are identified as momenta and $p_5 = m$ as mass. We have just

$$p_\mu\,\alpha^\mu\,\psi - m\psi = 0$$

and (5.3) becomes the energy-momentum-mass relation

$$p_\mu p^\mu = m^2$$

The other equations we shall study will be analogues of other massless theories and will reduce for $p_6 = 0$ to meson equations. Salam *et al.* (1965) have used a similar $SU(2,2)$ [equivalently $SO(4,2)$] formalism to attain a generalisation of the Bargmann–Wigner equations (Bargmann *et al.*, 1946); in these equations only the wave functions possess an $SO(4,2)$ transformation law, the equations themselves do not. The three equations we shall study are

(a) $$\bar{P}^{\bar{\alpha}\beta}\,\Xi_{\beta\gamma} = 0, \qquad \Xi_{\beta\gamma} + \Xi_{\beta\gamma} = 0 \tag{5.4}$$

(b) $$\bar{P}^{\bar{\alpha}\beta}\,\Xi_{\beta\gamma} = 0, \qquad \Xi_{\beta\gamma} - \Xi_{\beta\gamma} = 0 \tag{5.5}$$

(c) $$\bar{P}^{\bar{\alpha}\beta}\,\Xi_{\beta}^{\gamma} = 0, \qquad \Xi_{\alpha}^{\alpha} = 0 \tag{5.6}$$

The three 'wave functions' are irreducible under $SO(4,2)$ and have respectively 6, 10 and 15 components. They correspond respectively to a vector, an anti-self-dual rank three-skew tensor and a rank two-skew tensor, respectively:

(a) $$\Xi = \xi_A \sigma^A, \qquad \xi^A = \tfrac{1}{4} \text{ trace } \Xi\bar{\sigma}^A \tag{5.7}$$

(b) $$\Xi = (1/12)\, \xi_{ABC}\, \sigma^{ABC}, \qquad \xi^{ABC} = -\tfrac{1}{4} \text{ trace } \Xi\bar{\sigma}^{ABC} \tag{5.8}$$

(c) $$\Xi = \tfrac{1}{2}\xi_{AB}\, \sigma^{AB}, \qquad \xi^{AB} = -\tfrac{1}{4} \text{ trace } \Xi\sigma^{AB} \tag{5.9}$$

In terms of the tensors, the equations (5.4–5.6) become

(a) $$p_A \xi^A = 0, \qquad p_{[A}\, \xi_{B]} = 0 \tag{5.10}$$

(b) $$p_A \xi^{ABC} = 0, \qquad p_{[A}\, \xi_{BCD]} = 0 \tag{5.11}$$

(c) $$p_A \xi^{AB} = 0, \qquad p_{[A}\, \xi_{BC]} = 0 \tag{5.12}$$

Because of the anti-self-dual property of ξ_{ABC}, the two equations (b) are actually the same equation.

We now make the specialisation $p_6 = 0$, $p_5 = m$ and split the tensors as follows:

(a) $$\xi^A = (\xi^\mu, \xi^5, \xi^6) \tag{5.13}$$

(b) $$\left.\begin{aligned} \chi_\mu &= \xi_{\mu 56} = -\tfrac{1}{6} i \epsilon_{\mu\nu\rho\lambda}\, \xi^{\nu\rho\lambda} \\ \chi_{\mu\nu} &= \xi_{\mu\nu 6} = \tfrac{1}{2} i \epsilon_{\mu\nu\rho\lambda}\, \xi^{\rho\lambda 5} \end{aligned}\right\} \tag{5.14}$$

(c) $$\chi_{\mu\nu} = \xi_{\mu\nu}, \qquad \chi_\mu = \xi_{\mu 5}, \qquad \xi_\mu = \xi_{\mu 6}, \qquad \xi_5 = \xi_{56} \tag{5.15}$$

The equations (5.10) become the pseudoscalar meson equations

$$\left.\begin{aligned} p_\mu \xi^\mu &= m\xi_5 \\ p_\mu \xi_5 &= m\xi_\mu \\ \xi_6 &= 0 \end{aligned}\right\} \tag{5.16}$$

The equations (5.11) become the vector meson equations

$$p_\mu \chi^{\nu\mu} = m\chi^\nu, \qquad p_\mu \chi_\nu - p_\nu \chi_\mu = -m\chi_{\mu\nu} \tag{5.17}$$

$$p_\mu \chi^\mu = 0, \qquad p_{[\mu}\, \chi_{\nu\rho]} = 0 \tag{5.18}$$

and equation (5.12) gives the full set (5.16–5.18) (but of course without the trivial equation $\xi_6 = 0$ absent).

6. *The Kemmer Algebra*

The above analysis indicates a profound relationship between the algebra of Kemmer matrices (Kemmer, 1939, 1943) and the algebra of $SO(4,2)$. The specialisation to the (12346) subspace implicit in the restriction $p_s = m$ makes $\sigma^5 = \alpha^5$ into an invariant; we have, *for this subgroup*, a raising and lowering operator for spinor indices, $\sigma^5 C$. This corresponds to the fact that the basic spinor representations S and $\bar{S}$ are equivalent on the $SO(4,1)$ subgroup. Defining

$$\Gamma^\mu = \alpha^5 \sigma^\mu \qquad (\mu = 1, \ldots 4, 6) \tag{6.1}$$

(so that $\Gamma^\mu = \alpha^\mu$ ($\mu = 1,\ldots 4$) and $\Gamma^6 = \alpha^5$). Equations (5.4–5.6) can be rewritten (*without* the $p_6 = 0$ restriction).

$$\text{(a, b)} \qquad p_\mu \Gamma^{\mu}{}_{\alpha}^{\beta} \Xi_{\beta\gamma} + m\Xi_{\alpha\gamma} = 0 \tag{6.2}$$

$$\text{(c)} \qquad p_\mu \Gamma^{\mu}{}_{\alpha}^{\beta} \Xi^{\gamma}_{\beta} + m\Xi^{\gamma}_{\alpha} = 0 \tag{6.3}$$

Using the raising operator on the index γ of (6.2) we see immediately why equation (5.6), specialised to the subgroup, gave the same equations as (5.4) and (5.5).

Defining

$$\left.\begin{aligned} B^{(6)}_{\mu}{}^{B}_{A} &= \tfrac{1}{4}\,\text{trace}\,\bar{\sigma}_A \Gamma_\mu \sigma^B \qquad (\mu = 1234, 6) \\ B^{(10)}_{\mu}{}^{DEF}_{ABC} &= -\tfrac{1}{4}\,\text{trace}\,\bar{\sigma}_{ABC} \Gamma_\mu \sigma^{DEF} \\ B^{(15)}_{\mu}{}^{CD}_{AB} &= -\tfrac{1}{4}\,\text{trace}\,\sigma_{AB} \Gamma_\mu \sigma^{CD} \end{aligned}\right\} \tag{6.4}$$

we can write (a), (b) and (c) above as

$$\left.\begin{aligned} &\text{(a)} && p^\mu B^{(6)}_{\mu}{}^{B}_{A} \xi_B + m\xi_A = 0 \\ &\text{(b)} && \tfrac{1}{12} p^\mu B^{(10)}_{\mu}{}^{DEF}_{ABC} \xi_{DEF} + m\xi_{ABC} = 0 \\ &\text{(c)} && \tfrac{1}{2} p^\mu B^{(15)}_{\mu}{}^{CD}_{AB} \xi_{CD} + m\xi_{AB} = 0 \end{aligned}\right\} \tag{6.5}$$

If we regard the anti-self-dual or self-dual index set ABC as a single 10-fold index and the skew-symmetric pair as a single 15-fold index then $B^{(6)}_\mu$, $B^{(10)}_\mu$ and $B^{(15)}_\mu$ are sets of five matrices of dimension 6, 10 and 15. Making use of (4.9–4.11) and (6.4), by tedious calculation, we can verify that they satisfy the defining relations

$$B^\mu B^\nu B^\rho + B^\rho B^\nu B^\mu = g^{\mu\nu} B^\rho + g^{\rho\nu} B^\mu \tag{6.6}$$

of the five-dimensional Kemmer algebra. The commutators of pairs of these matrices are found to be

$$\left.\begin{aligned} &(6) && [B^\mu, B^\nu]^B_A = -\tfrac{1}{4}\,\text{trace}\,\bar{\sigma}_A \sigma^{\mu\nu} \sigma^B \\ &(10) && [B^\mu, B^\nu]^{DEF}_{ABC} = -\tfrac{1}{4}\,\text{trace}\,\bar{\sigma}_{ABC} \sigma^{\mu\nu} \sigma^{DEF} \\ &(15) && [B^\mu, B^\nu]^{CD}_{AB} = -\tfrac{1}{4}\,\text{trace}\,\sigma_{AB} \sigma^{\mu\nu} \sigma^{CD} \end{aligned}\right\} \tag{6.7}$$

$$(\mu, \nu = 12346)$$

If we convert the tensor indices A, ABC, AB in (6.7) to pairs of spinor indices $\alpha\beta$ (skew), $\alpha\beta$ (symmetric) and ${}^{\beta}_{\alpha}$ (traceless) by an obvious prescription, the right-hand sides of (6.7) become, when operating on a rank two spinor that is respectively skew-symmetric, symmetric and traceless, just a multiple of the generators of $SO(4,1)$ in a 'fusion' representation (the proof involves the use of (4.9–4.11)):

$$\sigma^{\mu\nu} \otimes 1 + 1 \otimes \sigma^{\mu\nu}$$

The irreducibility of the matrices $[B_\mu, B_\nu]$ then follows from the irreducibility of the $\sigma_{\mu\nu}$ in all three cases. Hence also the sets of five matrices B_μ are irreducible in all three cases. The constructions (6.4) are therefore expressions for the matrix elements of irreducible representations of the five-dimensional Kemmer algebra as traces products of Dirac matrices. Equations (6.5) are five-dimensional Kemmer equations, which, as we have seen, become the usual four-dimensional meson equations on setting $p_6 = 0$. The matrix set $B_\mu^{(6)}$ ($\mu = 1, \ldots 4$) is the direct sum of the one- and five-dimensional representations of the Kemmer algebra, $B_\mu^{(10)}$ is the ten-dimensional representation and $B_\mu^{(15)}$ is the direct sum of the five and the ten-dimensional representation.

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